The Quartic Formula (Euler)

We consider the quartic equation $x^4 + bx^3 + cx^2 + dx + e = 0$.

Let $x = z - b/4$. The equation reduces to: $z^4 + qz^2 + rz + s = 0$, where q, r, and $s \in \mathbf{R}$.

If $r = 0$, we can solve by the Quadratic Formula. Hence, assume $r \neq 0$.

(I) As with Descartes' Formula, we get the **resolvent cubic:**

$$j^3 + (2q)j^2 + (q^2 - 4s)j - r^2 = 0.$$

(II) Since there is one sign change, there is at least one positive root j_1. Let k_1 be its positive square root. Then j_1 has square roots $\pm k_1$.

(III) There are two other roots j_2 and j_3. They are either both real or complex conjugates. Each have square roots $\pm k_2$, $\pm k_3$.

(IV) By using the factorization $j^3 + (2q)j^2 + (q^2 - 4s)j - r^2 = (j - j_1)(j - j_2)(j - j_3)$, we have:

$$j_1 + j_2 + j_3 = -2q$$
$$j_1 j_2 j_3 = r^2.$$

Thus, depending on the signs of k_2 and k_3, we get $k_1k_2k_3 = \pm r$. There are eight possible choices for the signs of k_1, k_2, and k_3. We choose the four cases where:

$$k_1k_2k_3 = -r.$$

(V) Fix one of these cases. Then j_1 is the positive root of the resultant cubic. Let k_1 be its positive square root. We then have the factorization:

$$z^4 + qz^2 + rz + s = \left[z^2 + k_1 z + \frac{1}{2}\left(q + k_1^2 - \frac{r}{k_1}\right)\right]\left[z^2 - k_1 z + \frac{1}{2}\left(q + k_1^2 - \frac{r}{k_1}\right)\right].$$

(VI) We set the first factor equal to zero:

$$z^2 + k_1 z + \frac{1}{2}\left(q + k_1^2 - \frac{r}{k_1}\right) = 0$$

By the Quadratic Formula:

$$z = -\frac{k_1}{2} \pm \frac{1}{2}\sqrt{k_1^2 - 2\left(q + k_1^2 - \frac{r}{k_1}\right)}\ .$$

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(VII) The first formula from (IV) yields: $k_1^2 + k_2^2 + k_3^2 = -2q$. We have chosen $k_1k_2k_3$ so that their product is $-r$. Expanding inside the radical from (VI) and substituting, we get:

$$z = -\frac{k_1}{2} \pm \frac{1}{2}\sqrt{(k_2 - k_3)^2}\ .$$

(VII) This gives us two solutions:

$$\boxed{z_1 = \frac{1}{2}(-k_1 + k_2 - k_3); \qquad z_2 = \frac{1}{2}(-k_1 - k_2 + k_3)}$$

(VIII) We next set the second factor equal to zero.

$$z^2 - k_1 z + \frac{1}{2}\left(q + k_1^2 - \frac{r}{k_1} \right) = 0.$$

By similar methods as above, we obtain the other two solutions:

$$\boxed{z_3 = \frac{1}{2}(k_1 + k_2 + k_3); \qquad z_4 = \frac{1}{2}(k_1 - k_2 - k_3)}$$

(IX) To sum up, to solve the reduced quartic $z^4 + qz^2 + rz + s = 0$, we first find the roots j_1, j_2, and j_3 of the resolvent cubic $j^3 + (2q)j^2 + (q^2 - 4s)j - r^2 = 0$.

We choose their respective square roots k_1, k_2, and k_3 such that $k_1k_2k_3 = -r$. The roots of the reduced quartic are then:

$$\boxed{\begin{array}{ll} z_1 = \frac{1}{2}(-k_1 + k_2 - k_3); & z_2 = \frac{1}{2}(-k_1 - k_2 + k_3) \\ z_3 = \frac{1}{2}(k_1 + k_2 + k_3); & z_4 = \frac{1}{2}(k_1 - k_2 - k_3) \end{array}}$$

Subtracting $b/4$ from the roots give us the roots of the original equation:

$$x^4 + bx^3 + cx^2 + dx + e = 0.$$