The Quartic Formula (Descartes)

We consider the quartic equation $x^4 + bx^3 + cx^2 + dx + e = 0$.

Let $x = z - b/4$. The equation reduces to: $z^4 + qz^2 + rz + s = 0$, where q, r, and $s \in \mathbf{R}$.

If $r = 0$, we can solve by the Quadratic Formula or factoring. Hence, assume $r \neq 0$.

(I) Want k, k', l, and $m \in \mathbf{R}$ such that:

$$z^4 + qz^2 + rz + s = (z^2 + kz + l)(z^2 + k'z + m).$$

Expanding, we get:

$$z^4 + qz^2 + rz + s = z^4 + (k + k')z^3 + (m + kk' + l)z^2 + (mk + lk')z + lm = 0.$$

Since there is no x^3–term, we must have $k + k' = 0$, whence $k' = -k$. Thus:

$$z^4 + qz^2 + rz + s = z^4 + (m - k^2 + l)z^2 + (mk - lk)z + lm = 0.$$

(II) Equating coefficients, we have $q = m - k^2 + l$, $r = k(m - l)$, and $lm = s$.
Note that since $r \neq 0$, we must have $k \neq 0$.

(III) Since $k \neq 0$, $m - l = r/k$. Also, $m + l = q + k^2$.
Adding (and then subtracting) these two equations gives us:

$$2m = q + k^2 + \frac{r}{k}; \qquad 2l = q + k^2 - \frac{r}{k} \;;$$

So, once we determine k, we can obtain m and l.

(IV) Since $lm = s$, $(2l)(2m) = 4s$. From the above and doing some algebra:

$$\left(q + k^2 - \frac{r}{k}\right)\left(q + k^2 + \frac{r}{k}\right) = 4s$$

$$q^2 + 2k^2 q + k^4 - \frac{r^2}{k^2} = 4s$$

$$k^2 q^2 + 2k^4 q + k^6 - r^2 = 4sk^2$$

This gives us the equation:

$$k^6 + (2q)k^4 + (q^2 - 4s)k^2 - r^2 = 0.$$

The Quartic Formula (Descartes)

(V) Let $j = k^2$. We get the **resolvent cubic:**

$$j^3 + (2q)j^2 + (q^2 - 4s)j - r^2 = 0.$$

(VI) Since there is one sign change, there is at least one positive root (Descartes' Rule of Signs). Let $\pm k$ be its square roots. This will give us our desired factorization:

$$z^4 + qz^2 + rz + s = (z^2 + kz + l)(z^2 - kz + m) = \left[z^2 + kz + \frac{1}{2}\left(q + k^2 - \frac{r}{k} \right) \right] \left[z^2 - kz + \frac{1}{2}\left(q + k^2 - \frac{r}{k} \right) \right]$$

The roots of the factors give us the roots of $z^4 + qz^2 + rz + s = 0$.

Subtracting $b/4$ from the roots give us the roots of the original equation:

$$x^4 + bx^3 + cx^2 + dx + e = 0.$$