The Cubic Formula (Cardano)

We consider the quartic equation $x^3 + bx^2 + cx + d = 0$.

Let $x = z - b/3$. The equation reduces to: $z^3 + pz + q = 0$, where $p, q \in \mathbf{R}$.
Assume $p \neq 0$. (If $p = 0$, we get an easy case.)

(I) Let $z = A - \frac{p}{3A}$. Substituting into $z^3 + pz + q = 0$ and doing some algebra yields:

$$A^6 + qA^3 - \frac{p^3}{27} = 0 ,$$

which is quadratic in A^3. By the Quadratic Formula:

$$A^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad .$$

(II) We will take $A^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$. (The other case will come up shortly.)

(III) Thus A is <u>*any*</u> cube root of the right side. By abusing, the radical notation, we will write A as:

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \ .$$

(IV) We now look at the other case and relabel:

$$B^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \ .$$

Again, abusing the radical notation, we take its cube root:

$$B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

(V) Note that $A^3B^3 = -p^3/27$. Thus $AB = -p/3$, whence $B = -p/(3A)$. Thus by (I):

$$z = A - \frac{p}{3A} = A + B = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \quad .$$

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(VI) Note that A can be *any* cube root of $-\frac{q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}$. The other two cube roots are ωA and $\omega^2 A$, where $\omega = \frac{-1+i\sqrt{3}}{2}$ is the cube root of unity. Similarly, the cube roots of $-\frac{q}{2}-\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}$ are B, ωB, and $\omega^2 B$.

(VII) Since we have chosen the cube root A of $-\frac{q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}$ and so that the solution of the reduced equation $z^3 + pz + q = 0$ satisfies $z = A - p/(3A)$, the cube roots of $-\frac{q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}$ and $-\frac{q}{2}-\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}$ must have their product $= -p/3$.

Thus the roots are $z_1 = A + B$, $z_2 = \omega A + \omega^2 B$, and $z_3 = \omega^2 A + \omega B$.
(Note that $\omega^3 = 1$.)

(VIII) To recap, the solutions of the reduced equation $z^3 + pz + q = 0$ are:

$$z_1 = A + B,\ z_2 = \omega A + \omega^2 B, \text{ and } z_3 = \omega^2 A + \omega B,$$

where $A = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$ and $B = \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$.

Subtracting $b/3$ from the roots give us the roots of the original equation:

$$x^3 + bx^2 + cx + d = 0.$$