**The Cubic Formula (Cardano)**

We consider the cubic equation *x*3 + *bx*2 + *cx* + *d* = 0.

Let *x* = *z* – *b*/3. The equation reduces to: *z*3 + *pz* + *q* = 0, where *p*, *q* ∈ **R**.

Assume *p* ≠ 0. (If *p* = 0, we get an easy case.)

(I) Let . Substituting into *z*3 + *pz* + *q* = 0 and doing some algebra yields:

,

which is quadratic in *A*3. By the Quadratic Formula:

 .

(II) We will take . (The other case will come up shortly.)

(III) Thus *A* is *any* cube root of the right side. By abusing the radical notation, we will

write *A* as:

.

(IV) We now look at the other case and relabel:

.

Again, abusing the radical notation, we take its cube root:



(V) Note that *A*3*B*3 = –*p*3/27. Thus *AB* = –*p*/3, whence *B* = –*p*/(3*A*). Thus by (I):

 .

(VI) Note that *A* can be *any* cube root of . The other two cube roots

are *ωA* and *ω*2*A*, where  is the cube root of unity. Similarly, the

cube roots of  are *B*, *ωB*, and *ω*2*B*.

(VII) Since we have chosen the cube root *A* of  and so that the solution

of the reduced equation *z*3 + *pz* + *q* = 0 satisfies *z* = *A* – *p*/(3*A*), the cube roots of

 and  must have their product = –*p*/3.

Thus the roots are *z*1 = *A* + *B*, *z*2 = *ωA* + *ω*2*B*, and *z*3 = *ω*2*A* + *ωB*.

(Note that *ω*3 = 1.)

(VIII) To recap, the solutions of the reduced equation *z*3 + *pz* + *q* = 0 are:

*z*1 = *A* + *B*, *z*2 = *ωA* + *ω*2*B*, and *z*3 = *ω*2*A* + *ωB*,

where  and .

Subtracting *b*/3 from the roots give us the roots of the original equation:

*x*3 + *bx*2 + *cx* + *d* = 0.