Polynomial Equations

Luca Pacioli (1445–1517)

- In 1494, summarized all mathematical knowledge up to that time in his book *Summa de arithmetica, geometria , proportioni e proportionlità*.

www.biografiasyvidas.com

- Pacioli states: "It has not been possible until now to form general rules (to solve cubic and quartic equations)."

Scipione del Ferro (1465–1526)

- Served as Professor at the University of Bologna 1496–1526.

matematica.unibocconi.it

- Discovered how to solve a cubic of the form $x^3 + px = q$, where $p, q > 0$.

- Did not publish the result in his lifetime. He did share it with his colleague, Antonio Maria Fiore.

Niccolò Fontana (1499–1557)

- Venetian mathematician and an expert in military engineering.

- Acquired the nickname Tartaglia (the “Stutterer”) from a childhood injury.

- Bragged that he could solve cubics of the form $x^3 + mx^2 = n$.

http://venividivici.us

Fiore vs Tartaglia

- In 1535, Fiore challenges Tartaglia to a contest in which each person poses 30 problems. The loser pays for a banquet of 30 persons.

- All of Fiore's problems are solving cubics of the form $x^3 + px = q$, where $p, q > 0$.

- Tartaglia submitted problems of different types.

Fiore vs Tartaglia

- Tartaglia solves all of Fiore's problems the night before the deadline. Fiore can only solve some of Tartaglia's problems.

- Tartaglia declines the banquet.

- Fiore fades into obscurity.

Girolamo Cardano (1501–1576)

- Prominent scholar in medicine, mathematics, and philosophy.
- Loved to gamble and play chess.
- Spent most of his life in Milan.

www.pokertime.eu

Cardano & Tartaglia

- In 1539, Cardano inquires Tartaglia about his solution of the cubic. Tartaglia refuses.
- Cardano then invites Tartaglia to Milan to meet the military commander, Alfonso d'Avalos, so he can display his military inventions.
- Tartaglia reveals his cubic secret to Cardano, but makes Cardano swear an oath not to reveal it.

Cardano's Oath

March 29, 1539

"I swear to you, by the Sacred Gospel, and on my faith as a gentleman, not only never to publish your discoveries if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher, so that after my death no one shall be able to understand them."

The Aftermath

- Cardano discovers how to use Tartaglia's techniques to solve $x^3 = px + q$ and $x^3 + q = px$, where $p, q > 0$.

- The incessant squawking of a magpie one day assures Cardano of good luck. That same day, Ludovico Ferrari appears on his doorstep, looking for work. Cardano hires him as a servant.

Ludovico Ferrari (1522–1565)

- Showed exceptional talent. Becomes a “colleague” of Cardano before the age of 20.

- Cardano shared Tartaglia’s result with him.

- Had a reputation as a hothead and a brawler. Lost some fingers in a fight.

http://www.compendium.ro

Cardano's Result

- Cardano discovers how to solve the general cubic equation $x^3 + bx^2 + cx + d = 0$.

- The technique involves reducing to the reduced cubic $x^3 + px = q$ and using Tartaglia's result.

Ferrari's Result

- Ferrari discovers how to solve the general quartic equation $x^4 + bx^3 + cx^2 + dx + e = 0$.
- The technique involves reducing to the reduced cubic $x^3 + px = q$ and using Tartaglia's result.

The Loophole

- Cardano and Ferrari travel to Bologna and inspect the papers of Scipione del Ferro.
- They find, in del Ferro's own handwriting, the solution to the reduced cubic.
- Cardano reasons that he can now publish the results, since del Ferro discovered how to solve the reduced cubic.

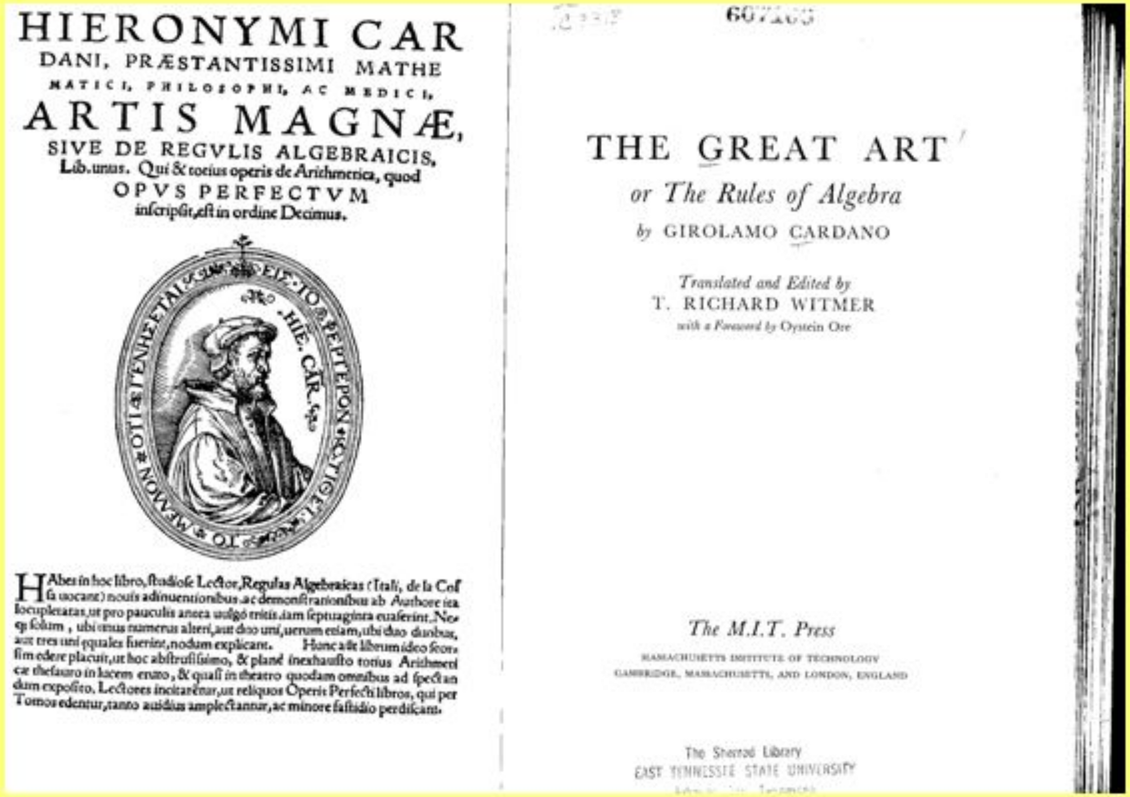
Cardano Publishes

- In 1545, Cardano publishes the *Ars Magna* (The Great Art).

- At the beginning of the book, he attributes the discovery of the solution to the cubic $x^3 + px = q$ to both del Ferro and Tartaglia.

- Cardano reveals the solution in Chapter XI, once again crediting del Ferro and Tartaglia.

Ars Magna



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aűt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

THE GREAT ART
or The Rules of Algebra
by GIROLAMO CARDANO

Translated and Edited by
T. RICHARD WITMER
with a Foreword by Oystein Ore

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maa.org

Ars Magna

PROBLEM XII

If someone says

$$x^4 + 3 = 12x,$$

add $2bx^2 + b^2$. It is clear that, without [3] the constant, this would have a square root. So we add $2bx^2$ to the other side and, as a constant, $b^2 - 3$. You will have these parts: $x^4 + 2bx^2 + b^2$, and $2bx^2 + 12x + b^2 - 3$. Therefore having multiplied the parts,[42] you have

$$2b^3 = 6b + 36$$

and

$$b^3 = 3b + 18,$$

wherefore b equals 3. Hence the parts are $x^4 + 6x^2 + 9$ and $6x^2 + 12x + 6$, and $x^2 + 3$, the square root of the first, will be equal to $x\sqrt{6} + \sqrt{6}$, and the value of x will be $\sqrt{\sqrt{6} - 1\frac{1}{2}} + \sqrt{1\frac{1}{2}}$.

Tartaglia vs Cardano

http://venividivici.us

www.pokertime.eu

Tartaglia accuses Cardano of breaking his oath.

Tartaglia vs Ferrari

http://venividivici.us

http://www.compendium.ro

Ferrari takes up Cardano's cause. Both sides write nasty letters to each other.

Tartaglia vs Ferrari

Ferrari writes Tartaglia in 1547, describing him as:

"*…someone who spends the whole time … on trifles.*
I promise you that if it were up to me to reward you,
I would load you up so much with roots and radishes,
that you would never eat anything else in your life."

Tartaglia vs Ferrari

- Tartaglia and Ferrari meet for a debate in Milan on August 10, 1548. Milan is Ferrari's hometown.
- Cardano does not attend. Tartaglia accuses him of cowardice.
- Ferrari is declared the winner of the debate. Tartaglia does not stay to hear the result.

Cardano's Formula

$$f(x) = x^3 + bx^2 + cx + d = 0$$

Letting $x = y - b/3$ gives us the reduced case:

$$y^3 + py + q = 0$$

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} = A + B;$$

$$y = \omega A + \omega^2 B; \quad y = \omega^2 A + \omega B; \text{ where } \omega = \frac{-1 + i\sqrt{3}}{2}.$$

The Discriminant

Let $\Delta = \frac{p^3}{27} + \frac{q^2}{4}$.

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} = A + B;$$

$$y = \omega A + \omega^2 B; \quad y = \omega^2 A + \omega B.$$

- If $\Delta > 0$, one real and two complex solutions.
- If $\Delta = 0$, all real roots, some repeated.
- If $\Delta < 0$, three real distinct solutions. (Irreducible Case)

René Descartes (1596–1650)

- Philosopher, scientist, and mathematician.
- Had dreams in 1619 which led him to find a new basis for scientific inquiry.
- Displayed his solution to the quartic in *La Géométrie*.

en.wikipedia.org/

Descartes' Quartic Formula

Reduced Quartic: $f(z) = z^4 + qz^2 + rz + s = 0$

Assume $r \neq 0$ and $q^2 - 4s < 0$.

Resolvent Cubic: $y^3 + (2q)y^2 + (q^2 - 4s)y - r^2 = 0.$

Let k be the positive square root of a positive real root of the resolvent cubic. (Since $q^2 - 4s < 0$, there is such a root.)

Descartes' Quartic Formula

Then $f(z) = z^4 + qz^2 + rz + s$ factors as:

$$\left[z^2 + kz + \frac{1}{2}\left(q + k^2 - \frac{r}{k}\right)\right]\left[z^2 - kz + \frac{1}{2}\left(q + k^2 - \frac{r}{k}\right)\right].$$

The roots of $f(z)$ can then be computed.

René Descartes

Closing remark of *La Géométrie*:

"*I hope that posterity would judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted, so as to leave to the others the pleasure of discovery.*"

Leonhard Euler (1707–1783)

- Mathematician and physicist.
- In Chapter XII of *Elements of Algebra*, Euler rederives Cardano's Formula using a quadratic.
- In Chapter XV, Euler derives his Quartic Formula.

http://www.afaqmaroc.com

Euler's Quartic Formula

Reduced Quartic: $f(z) = z^4 + qz^2 + rz + s = 0; \; r \neq 0$

Resolvent Cubic: $y^3 + (2q)y^2 + (q^2 - 4s)y - r^2 = 0.$
Let j_1, j_2, j_3 be its roots.

Solutions: The four sums $\frac{1}{2}(k_1 + k_2 + k_3)$, where k_1, k_2, and k_3 are the respective square roots of j_1, j_2, j_3, chosen so that:

$$k_1 k_2 k_3 = -r.$$