Cubic Equation

Solve: $y^3 + 6y = 20$
$y^3 + 6y - 20 = 0$; $p = 6$, $q = -20$ (Already in the reduced form.)

Discriminant: $\Delta = \frac{p^3}{27} + \frac{q^2}{4} = 108 > 0$; One real, two complex roots.

First root:

$$y_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$
$$= \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$$

Both terms are elements of $\mathbf{Q}\left(\sqrt{3}\right) = \left\{a + b\sqrt{3} \middle| a, b \in \mathbf{R}\right\}$.

$$\sqrt[3]{10 + 6\sqrt{3}} = a + b\sqrt{3}$$
$$10 + 6\sqrt{3} = \left(a + b\sqrt{3}\right)^3$$
$$10 + 6\sqrt{3} = (a^3 + 9ab^2) + (3a^2b + 3b^3)\sqrt{3}$$

We solve the system:

$$a^3 + 9ab^2 = 10$$
$$3a^2b + 3b^3 = 6.$$

That is:

$$a(a^2 + 9b^2) = 10$$
$$b(a^2 + b^2) = 2.$$

This is true when $a = b = 1$. Thus $\sqrt[3]{10 + 6\sqrt{3}} = 1 + \sqrt{3}$. Similarly $\sqrt[3]{10 - 6\sqrt{3}} = 1 - \sqrt{3}$.
This yields: $y_1 = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = \left(1 + \sqrt{3}\right) + \left(1 - \sqrt{3}\right) = \boxed{2}$.

Other two solutions:

$$y_2 = \omega A + \omega^2 B = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(1 + \sqrt{3}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(1 - \sqrt{3}\right) = \boxed{-1 + 3i}$$

$$y_3 = \omega^2 A + \omega B = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(1 + \sqrt{3}\right) + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(1 - \sqrt{3}\right) = \boxed{-1 - 3i}$$

Solutions: $\boxed{y = 2, -1 \pm 3i}$

Cubic Equation

<u>Solve:</u> $f(x) = x^3 - 3x^2 - 4x + 12 = 0$; Let $x = y - b/3 = y + 1$.

<u>Reduced Form:</u> $f(y + 1) = y^3 - 7y + 6$; $p = -7$, $q = 6$

<u>Discriminant:</u> $\Delta = \frac{p^3}{27} + \frac{q^2}{4} = -\frac{100}{27} < 0$; Three real distinct roots.

<u>First root:</u>

$$y_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

$$= \sqrt[3]{-3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{-3 - \sqrt{-\frac{100}{27}}} = \sqrt[3]{-3 + \frac{10}{9} i\sqrt{3}} + \sqrt[3]{-3 - \frac{10}{9} i\sqrt{3}}$$

Both terms are elements of $\mathbf{Q}\left(\sqrt{3}\right) = \left\{a + b\sqrt{3} \middle| a, b \in \mathbf{R}\right\}$.

$$\sqrt[3]{-3 + \frac{10}{9} i\sqrt{3}} = a + bi\sqrt{3}$$

$$-3 + \frac{10}{9} i\sqrt{3} = \left(a + bi\sqrt{3}\right)^3$$

$$-3 + \frac{10}{9} i\sqrt{3} = (a^3 - 9ab^2) + (3a^2b - 3b^3)i\sqrt{3}$$

We solve the system:

$$a^3 - 9ab^2 = -3$$
$$3a^2b - 3b^3 = 10/9.$$

That is:

$$a(a^2 - 9b^2) = -3$$
$$b(a^2 - b^2) = 10/27.$$

This is true when $a = 1$ and $b = 2/3$.

Thus $-3 + \frac{10}{9} i\sqrt{3} = 1 + \frac{2}{3} i\sqrt{3}$. Similarly $-3 - \frac{10}{9} i\sqrt{3} = 1 - \frac{2}{3} i\sqrt{3}$.

This yields: $y_1 = \sqrt[3]{-3 + \frac{10}{9} i\sqrt{3}} + \sqrt[3]{-3 - \frac{10}{9} i\sqrt{3}} = \left(1 + \frac{2}{3} i\sqrt{3}\right) + \left(1 - \frac{2}{3} i\sqrt{3}\right) = \boxed{2}$.

Other two solutions:

$$y_2 = \omega A + \omega^2 B = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(1 + \frac{2}{3}i\sqrt{3}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(1 - \frac{2}{3}i\sqrt{3}\right) = \boxed{-3}$$

$$y_3 = \omega^2 A + \omega B = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(1 + \frac{2}{3}i\sqrt{3}\right) + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(1 - \frac{2}{3}i\sqrt{3}\right) = \boxed{1}$$

Thus $x_1 = 2 + 1 = 3$; $x_2 = -3 + 1 = -2$; $x_3 = 1 + 1 = 2$

Solutions: $\boxed{x = 3, \pm 2}$

Quartic Equation (Ferrari)

Solve: $z^4 - 12z + 3 = 0$.

First, we do some algebra.

$$
\begin{aligned}
z^4 &= 12z - 3 \\
z^4 + 2z^2t + t^2 &= 12z - 3 + 2z^2t + t^2 \\
(z^2 + t)^2 &= 12z - 3 + 2z^2t + t^2 \\
(z^2 + t)^2 &= 12z - 3 + 2z^2t + t^2
\end{aligned}
$$

Want to find a t such that the right side $(2t)z^2 + 12z + (t^2 - 3)$ is a perfect square. This is a quadratic in z. It is a perfect square if its discriminant $144 - 4(2t)(t^2 - 3) = 0$. This yields the cubic:

$$8t^3 - 24t - 144 = 0$$

or

$$t^3 - 3t - 18 = 0.$$

By Cardano's Formula, with $p = -3$ and $q = -18$:

$$t = \sqrt[3]{9+4\sqrt{5}} + \sqrt[3]{9-4\sqrt{5}} \quad .$$

Both terms are elements of $\mathbf{Q}\left(\sqrt{5}\right) = \left\{a + b\sqrt{5} \middle| a, b \in \mathbf{R}\right\}$.

$$
\begin{aligned}
\sqrt[3]{9+4\sqrt{5}} &= a + b\sqrt{5} \\
9 + 4\sqrt{5} &= \left(a + b\sqrt{5}\right)^3 \\
9 + 4\sqrt{5} &= (a^3 + 15ab^2) + (3a^2b + 5b^3)\sqrt{5}
\end{aligned}
$$

We solve the system:

$$
\begin{aligned}
a^3 + 15ab^2 &= 9 \\
3a^2b + 5b^3 &= 4.
\end{aligned}
$$

That is:

$$
\begin{aligned}
a(a^2 + 15b^2) &= 9 \\
b(3a^2 + 5b^2) &= 4.
\end{aligned}
$$

This is true when $a = 3/2$ and $b = \pm 1/2$.

Thus $t = \sqrt[3]{9+4\sqrt{5}} + \sqrt[3]{9-4\sqrt{5}} = \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) + \left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right) = 3$.

So we get $(z^2 + 3)^2 = 6z^2 + 12z + 6 = 6(z + 1)^2$, whence $z^2+3=\pm\sqrt{6}(z+1)$.

This gives the two quadratic equations:

$$z^2 - z\sqrt{6} + \left(3-\sqrt{6}\right) = 0$$
$$z^2 + z\sqrt{6} + \left(3+\sqrt{6}\right) = 0$$

By the Quadratic Formula:

$$\boxed{z = \frac{\sqrt{6} \pm \sqrt{4\sqrt{6}-6}}{2},\quad \frac{-\sqrt{6} \pm i\sqrt{6+4\sqrt{6}}}{2}}$$

Quartic Equation (Déscartes)

Solve: $x^4 - 4x^2 - 8x + 35 = 0.$ $\quad q = -4,\ r = -8,\ s = 35$

Associated Cubic: $j^3 + (2q)j^2 + (q^2 - 4s)j - r^2 = 0$

$j^3 - 8j^2 - 124j - 64 = 0$

A positive root of the cubic is $j = 64$. So let $k = 8$.

We can then factor the original polynomial as:

$$x^4 - 4x^2 - 8x + 35 = \left[x^2 + kx + \frac{1}{2}\left(q + k^2 - \frac{r}{k}\right)\right]\left[x^2 - kx + \frac{1}{2}\left(q + k^2 + \frac{r}{k}\right)\right]$$
$$= (x^2 + 4x + 7)(x^2 - 4x + 5)$$

Solutions: $\boxed{x = -2 \pm \sqrt{3},\ 2 \pm i}$

Quartic Equation (Euler)

Solve: $x^4 - 25x^2 + 60x - 36 = 0$. $q = -25$, $r = 60$, $s = -36$

Associated Cubic: $j^3 + (2q)j^2 + (q^2 - 4s)j - r^2 = 0$
$j^3 - 50j^2 + 769j - 3600 = 0$

Roots of associated cubic: $j_1 = 9$, $j_2 = 16$, $j_3 = 25$, whence $k_1 = \pm 3$, $k_2 = \pm 4$, $k_3 = \pm 5$.
We choose the k's so that $k_1k_2k_3 = -60$.

Roots

$$x_1 = \frac{1}{2}(3+4-5) = 1; \quad x_2 = \frac{1}{2}(3-4+5) = 2$$

$$x_3 = \frac{1}{2}(-3+4+5) = 3; \quad x_4 = \frac{1}{2}(-3-4-5) = -6$$

Solutions: $\boxed{x = 1, 2, 3, -6}$